Eigenvalues and Eigenvectors

Aim:

To construct strong knowledge in theory of Matrix and its application of matrices

Objectives:

Introducing the basic concepts of Eigenvalues and Eigenvectors

Prerequisites:

Students should have basic knowledge in the concepts and applications of matrices.

Learning Outcome:

Students should be able to do basic computation of Eigenvalues and Eigenvectors

Recall

- Matrix
- Order of a Matrix
- Determinant
- Transpose of a Matrix
- Identity Matrix
- Multiplication of two Matrices
- Inverse of a Matrix
- Symmetric and Non-symmetric Matrix
- Singular and Non-singular Matrix

Outline

Eigenvalues and Eigenvectors

- Introduction
- Definition of Matrix and various terms
- Characteristic Equation of a matrix
- Properties of Eigenvalues
- Finding Eigenvalues and Eigenvectors

Eigenvalues and Eigenvectors

Definition: Matrix

A system of mn numbers(elements) arranged in a rectangular arrangement along m rows and n columns and bounded by the brackets [ ] or ( ) is called an m by n matrix, which is written as $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & .. & .. & a_{1n} \\ a_{21} & a_{22} & .. & .. & a_{2n} \\ \ldots & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ a_{m1} & a_{m2} & .. & .. & a_{mn} \end{bmatrix}$$

Characteristic polynomial

The determinant $|A - \lambda I|$ when expanded will give a polynomial, which we call as characteristic polynomial of matrix A.

Definition: Eigenvalues

A = [a_{ij}] be a square matrix.

The characteristic equation of A is $|A - \lambda I| = 0$.

The roots of the characteristic equation are called Eigenvalues of A.

<u>Definition:</u> Eigenvectors

A = [a_{ij}] be a square matrix of order 'n'

If there exist a non zero vector X = $\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$

such that AX = λX , then the vector X is called an Eigenvector of A corresponding to the Eigenvalue λ .

Method of finding characteristic equation of a 3x3 matrix and 2x2 matrix

The characteristic equation of a 3x3 matrix is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S1= sum of main diagonal elements.

S_2 = sum of minor of main diagonal elements.

S_3 = Det (A) =|A|

The characteristic equation of a 2x2 matrix is $\lambda^2 - S_1\lambda + S_2 = 0$

Where, S_1 = sum of main diagonal elements.

S_2 = Det (A) =|A|

1. Find the characteristic equation of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^2 - S_1\lambda + S_2 = 0$

S_1 = sum of main diagonal elements

= 1+2=3

S_2 = Det (A) =|A|

$= \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix}$

S_2 = 2-0 = 2

The characteristic equation is $\lambda^2 - 3\lambda + 2 = 0$.

2. Find the characteristic equation of $\begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1 = sum of the main diagonal elements

= 2+1-4 = -1

S_2 = sum of minor of main diagonal elements

$= \begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -5 & -4 \end{vmatrix} + \begin{vmatrix} 2 & -3 \\ 3 & 1 \end{vmatrix}$

= (-4-6)+(-8+5)+(2+9) = -10+(-3)+11 = -2

S_3 = Det (A) =|A|

$= \begin{vmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{vmatrix}$

= 2(-4-6)-(-3)(-12+15)+1(6+5)

= 2(-10) +3(3) +1(11) = -20+9+11= 0

The characteristic equation is $\lambda^3 + \lambda^2 - 2\lambda = 0$

3. Find the Eigenvalues of $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Solution:

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$S_1 = 1+2+1=4$$

$$S_2 = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= (2\text{-}1)+(1\text{-}0)+(2\text{-}1) = 3$$

$$S_3 = \begin{vmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1(2\text{-}1)+(\text{-}1\text{-}0)+0 = 0$$

Therefore the characteristic equation is $\lambda^3 - 4\lambda^2 + 3\lambda - 0 = 0$

To find the Eigenvalues

$$\lambda^3 - 4\lambda^2 + 3\lambda - 0 = 0$$

$$\lambda(\lambda^2 \text{-}4\lambda\text{+}3) = 0$$

$$\lambda = 0,\ (\lambda^2 \text{-}4\lambda\text{+}3) = 0$$

$$(\lambda-1)(\lambda-3) = 0$$

$$\lambda = 1, 3$$

The Eigen values are 1, 3, and 0.

Properties of Eigenvalues.

i. The sum of the Eigenvalues of a matrix is the sum of the elements of main diagonal
ii. The product of the Eigenvalues is equal to the determinant of the matrix.
iii. The Eigen values of the triangular matrix are just the diagonal element of the matrix
iv. If λ is an Eigenvalue of a matrix A, then $1/\lambda$, ($\lambda \neq 0$) is Eigen value of A^{-1}
v. If $\lambda_1, \lambda_2, \ldots\ldots\ldots \lambda_n$ are the Eigenvalues of a matrix A, then A^m has a Eigenvalues $\lambda_1^m, \lambda_2^m, \ldots\ldots \lambda_n^m$

1. Find the sum & product of the Eigenvalues of the matrix $A=\begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & -6 \end{bmatrix}$

Solution:

Sum of the Eigen values = sum of the main diagonal elements

= 2+3-6=-1

Product of the Eigen value = |A|

$$= 2\begin{vmatrix} 3 & 1 \\ 1 & -6 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & -6 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}$$

= 2(-18-1)-1(-6-2)+2(1-6) = - 40

Sum = -1 and Product = - 40

2. The product of two Eigenvalues of the matrix $A=\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ **is 16. Find the third Eigenvalue of A.**

Solution:

Given: The product of two Eigen values of A is 16

(i.e) $\lambda_1\lambda_2 = 16$

By property, Product of Eigen values = |A|

$$\lambda_1\lambda_2\lambda_3 = |A|$$

$$16\lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9-1)+2(-6+2)+2(2-6)= 32$$

$$\lambda_3 = \frac{32}{16} = 2$$

The third Eigen value is 2.

3. Two Eigenvalues of the matrix A = $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ **are 3 and 0. What is the third eigenvalue?**

What is the product of the eigenvalues of A?

Solution:

Given: If $\lambda_1 = 3$, $\lambda_2 = 0$, and $\lambda_3 =$?

By property, Sum of the Eigenvalues = sum of the main diagonals.

$$\lambda_1+\lambda_2+\lambda_3 = 8+7+3= 18$$

$$3+0+\lambda_3 = 18$$

$$\lambda_3 = 18 - 3 = 15$$

By property, Product of the Eigen values = |A|

$$(3)(0)(15) = |A|$$

$$|A| = 0$$

The third eigenvalue is 15, The product of the eigenvalues of A is 0.

4. If 3 and 15 are two Eigenvalues of the matrix A = $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ **then find the third eigenvalue and hence** = |A|

Solution:

Given: If $\lambda_1 = 3$, $\lambda_2 = 15$, and $\lambda_3 = ?$

By property, Sum of the Eigenvalues = sum of the main diagonals.

$$\lambda_1+\lambda_2+\lambda_3 = 8+7+3= 18$$

$$3+15+\lambda_3 = 18$$

$$\lambda_3 = 18 - 18 = 0$$

By property, Product of the Eigenvalues = |A|

$$(3)(15)(0) = |A|$$

$$|A| = 0$$

The third eigenvalue is 0, The product of the eigenvalues of A is 0.

Non-Symmetric Matrix With Non-Repeated Eigenvalues

1. Find all the Eigenvalues and Eigenvectors of the matrix $\begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$

Solution:

Given: $A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$

To find the characteristic equation of A

Formula: The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1= sum of main diagonal

=1+2-1=2

S_2 = sum of minor of main diagonal elements

$$= \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix}$$

= (-2+1)+(-1-8)+(2+3) = -1-9+5 = -5

S_3 = Det (A)=|A|

$$=\begin{vmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{vmatrix} = 1(-2+1)+1(-3+2)+4(3-4)$$

$$=1(-1)+1(-1)+4(-1) = -1-1-4= -6$$

Hence the characteristic equation is $\lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

To solve the characteristic equation:

If $\lambda=1$ By synthetic division

1	1	-2	-5	6
	0	1	-1	- 6
	1	-1	- 6	0

Therefore the $\lambda=1$ and other roots are given by $\lambda^2 - \lambda - 6 = 0$

$$(\lambda+2)(\lambda-3) = 0$$

$$\lambda = -2, 3$$

Therefore Eigenvalues are 1,-2, 3

To find the Eigenvectors:

To get the Eigenvectors solve: $(A-\lambda I)X=0$

$$\left[\begin{pmatrix}1 & -1 & 4\\ 3 & 2 & -1\\ 2 & 1 & -1\end{pmatrix} - \lambda\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right]\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$\left[\begin{pmatrix}1-\lambda & -1 & 4\\ 3 & 2-\lambda & -1\\ 2 & 1 & -1-\lambda\end{pmatrix}\right]\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$\left.\begin{aligned}(1-\lambda)x_1-x_2+4x_3&=0\\ 3x_1+(2-\lambda)x_2-x_3&=0\\ 2x_1+x_2+(-1-\lambda)x_3&=0\end{aligned}\right\} \quad \ldots.(1)$$

Case 1: Substitute $\lambda=1$ in, (1) we get

$$0x_1-x_2+4x_3=0 \quad \ldots. \quad (2)$$
$$3x_1+x_2-x_3=0 \quad \ldots. \quad (3)$$
$$2x_1+x_2-2x_3=0 \quad \ldots. \quad (4)$$

Solving (2) and (3) by cross multiplication rule, we get

	x_1	x_2	x_3
-1	4	0	-1
1	-1	3	1

$$\frac{x_1}{1-4} = \frac{x_2}{12-0} = \frac{x_3}{0+3}$$

$$\Rightarrow \frac{x_1}{-3} = \frac{x_2}{12} = \frac{x_3}{3}$$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_2}{4} = \frac{x_3}{1}$$

Therefore $X_1 = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}$

Case 2: Substitute λ= -2 in (1), we get

$3x_1 - x_2 + 4x_3 = 0$ ....(5)

$3x_1 + 4x_2 - x_3 = 0$ ....(6)

$2x_1 + x_2 + x_3 = 0$ ....(7)

Solving (5) and (6) by cross multiplication rule we get

	x_1		x_2		x_3	
-1		4		3		-1
4		-1		3		4

$$\frac{x_1}{1-16} = \frac{x_2}{12+3} = \frac{x_3}{12+3}$$

$$\Rightarrow \frac{x_1}{-15} = \frac{x_2}{15} = \frac{x_3}{15}$$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1}$$

Therefore $X_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

Case 3: Substitute $\lambda=3$ in (1) we get

$-2x_1-x_2+4x_3=0$ …. (8)

$3x_1-x_2-x_3=0$ …. (9)

$2x_1+x_2-4x_3=0$ …. (10)

Solving (8) and (9) by cross multiplication rule we get

x_1		x_2		x_3	
-1	4		-2	-1	
-1	-1		3	-1	

$$\frac{x_1}{1+4} = \frac{x_2}{12-2} = \frac{x_3}{2+3}$$

$$\Rightarrow \frac{x_1}{5} = \frac{x_2}{10} = \frac{x_3}{5}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

Therefore $X_3=\begin{pmatrix}1\\2\\1\end{pmatrix}$

Result: The Eigen values of A are 1,-2, 3 and the Eigenvectors are $\begin{pmatrix}-1\\4\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\1\end{pmatrix}$

Non-Symmetric Matrix With Repeated Eigenvalues

1. Find all the Eigenvalues and Eigenvectors of the matrix $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

Solution:

Given: A= $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

To find the characteristic equation of A

Formula: The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1 = sum of main diagonal

= -2+1+0=-1

S_2 = sum of minor of main diagonal elements

$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$

= (0-12)+(0-3)+(-2-4) = -12-3-6=-21

S_3 = Det (A)=|A|

$= \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$ = -2(0-12)-2(0-6)-3(-4+1) =45

Hence the characteristic equation is $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

if λ=1; 1+1-21-45 ≠0
if λ=-1; -1+1-21-45 ≠0
if λ=2; 8+4-42-45 ≠0
if λ=-2; -8+4+42-45≠0
if λ=3; 27+9-63-45≠0
if λ=-3; -27+9+63-45≠0

Therefore λ=-3 is a root

By synthetic division

-3	1	1	-21	-45
	0	-3	6	45
	1	-2	-15	0

Therefore the λ= -3 and other roots are given by $\lambda^2 - 2\lambda - 15 = 0$

(λ-5)(λ+3) = 0

λ= 5,-3,-3

Therefore Eigenvalues are 5, -3,-3 and Here the Eigenvalues are repeated.

To find the Eigenvectors:

To get the Eigenvectors solve (A-λI)X=0

$$\left[\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned}(-2-\lambda)x_1+2x_2-3x_3=0\\ 2x_1+(1-\lambda)x_2-6x_3=0\\ -x_1-2x_2+-\lambda x_3=0\end{aligned}\right\} \dots \qquad (1)$$

Case 1: Substitute $\lambda=5$ in (1) we get

$$-7x_1+2x_2-3x_3=0 \quad \dots.(2)$$

$$2x_1-4x_2-6x_3=0 \quad \dots.(3)$$

$$-x_1-2x_2-5x_3=0 \quad \dots.(4)$$

Solving (3)and (4) by cross multiplication rule we get

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -4 & -6 & 2 & -4 \\ -2 & -5 & -1 & -2 \end{array}$$

$$\frac{x_1}{20-12}=\frac{x_2}{6+10}=\frac{x_3}{-4-4}$$

$$\Rightarrow \frac{x_1}{8}=\frac{x_2}{16}=\frac{x_3}{-8} \quad \Rightarrow \frac{x_1}{1}=\frac{x_2}{2}=\frac{x_3}{-1}$$

Therefore $X_1=\begin{pmatrix}1\\2\\-1\end{pmatrix}$

Case 2: Substitute $\lambda=-3$ in (1), we get

$$x_1+2x_2-3x_3=0 \quad \dots.\ (5)$$

$$2x_1+4x_2-6x_3=0 \quad \dots.\ (6)$$

$$x_1+2x_2-3x_3=0 \quad \dots.\ (7)$$

Since (5),(6),(7) are all same, So we considered only one equation

$$x_1+2x_2-3x_3=0$$

Put $x_1=0$

$$2x_2-3x_3=0$$

$$\Rightarrow 2x_2=3x_3$$

$$\frac{x_2}{3}=\frac{x_3}{2}$$

Therefore Eigenvector is $X_2=\begin{pmatrix}0\\3\\2\end{pmatrix}$

Put $x_2=0$

$$x_1-3x_3=0$$

$$\Rightarrow x_1=3x_3$$

$$\frac{x_1}{3}=\frac{x_3}{1}$$

Therefore Eigenvector is $X_3=\begin{pmatrix}3\\0\\1\end{pmatrix}$

Result: The Eigenvalues are-3,-3,5 and Eigenvectors are $\begin{pmatrix}-1\\-2\\1\end{pmatrix},\begin{pmatrix}0\\3\\2\end{pmatrix},\begin{pmatrix}3\\0\\1\end{pmatrix}$

Symmetric Matrix With Non-Repeated Eigenvalues

1. Find all the Eigen values and Eigen vectors of the matrix $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

Solution:

Given: $A=\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$

To find the characteristic equation of A

Formula: The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where,

S_1 = sum of main diagonal

$=1+5+1=7$

S_2 = sum of minor of main diagonal elements

$= \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix} = (5-1)+(1-9)+(5-1)=0$

$S_3 = |A| = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix} = 1(5-1)-1(1-3)+3(1-15) = -36$

Hence the characteristic equation is $\lambda^3 - 7\lambda^2 + 0\lambda - 36 = 0$

if $\lambda=1$; $1-7+0+36 \neq 0$

if $\lambda=-1$; $-1-7+0+36 \neq 0$

if $\lambda=2$; $8-24+0+36 \neq 0$

if $\lambda=-2$; $-8-24+0+36=0$

$\lambda = -2$ is a root

To solve the characteristic equation:

if λ=-2 By synthetic division

-2	1	-7	0	36
	0	-2	18	-36
	1	-9	18	0

Therefore the λ=-2 and other roots are given by $\lambda^2 - 9\lambda + 18 = 0$

$(\lambda-6)(\lambda-3) = 0$

$\lambda = 3, 6$

Therefore Eigenvalues are-2, 3, 6

To find the Eigenvectors:

To get the Eigenvectors solve (A-λI)X=0

$$\left[\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{r} (1-\lambda)x_1+x_2+3x_3=0 \\ x_1+(5-\lambda)x_2+x_3=0 \\ 3x_1+x_2+(1-\lambda)x_3=0 \end{array}\right\} \quad \ldots(1)$$

Case 1: Substitute λ=-2 in (1), we get

$3x_1+x_2+3x_3=0$ …(2)

$x_1+7x_2+x_3=0$ …(3)

$3x_1+x_2+3x_3=0$ …(4)

Since (2) and (4) are same we consider, solving (2)and (3) by cross multiplication rule we get

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & 3 & 3 & 1 \\ 7 & 1 & 1 & 7 \end{array}$$

$$\frac{x_1}{1-21} = \frac{x_2}{3-3} = \frac{x_3}{21-1}$$

$$\Rightarrow \frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20} \Rightarrow \frac{x_1}{-10} = \frac{x_2}{0} = \frac{x_3}{10}$$

Therefore $X_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

Case 2: Substitute $\lambda=3$ in (1), we get

$$-2x_1+x_2+3x_3=0 \quad \dots (5)$$

$$x_1+2x_2+x_3=0 \quad \dots (6)$$

$$3x_1+x_2-2x_3=0 \quad \dots (7)$$

Solving (5) and (6) by cross multiplication rule we get

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & 3 & -2 & 1 \\ 2 & 1 & 1 & 2 \end{array}$$

$$\frac{x_1}{1-6} = \frac{x_2}{2+3} = \frac{x_3}{-4-1}$$

$$\Rightarrow \frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

Therefore $X_2=\begin{pmatrix}-1\\1\\-1\end{pmatrix}$

Case 3: Substitute λ=6 in (1), we get

$$-5x_1+x_2+3x_3=0 \quad \ldots (8)$$

$$x_1-x_2+x_3=0 \quad \ldots (9)$$

$$3x_1+x_2-5x_3=0 \quad \ldots (10)$$

	x_1		x_2		x_3	
1		3		-5		1
-1		1		1		-1

$$\frac{x_1}{1+3}=\frac{x_2}{3+5}=\frac{x_3}{5-1}$$

$$\Rightarrow \frac{x_1}{4}=\frac{x_2}{8}=\frac{x_3}{4} \qquad \Rightarrow \frac{x_1}{1}=\frac{x_2}{2}=\frac{x_3}{1}$$

Therefore $X_3=\begin{pmatrix}1\\2\\1\end{pmatrix}$

Therefore the Eigenvalues of A are 6,-2, 3

Result: The Eigenvalues of A are 6,-2, 3 and the Eigenvectors are $\begin{pmatrix}-1\\0\\1\end{pmatrix}$, $\begin{pmatrix}-1\\1\\-1\end{pmatrix}$, $\begin{pmatrix}1\\2\\1\end{pmatrix}$

Symmetric Matrix With Repeated Eigenvalues

1. Find all the Eigenvalues and Eigenvectors of $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

Solution:

Given: A= $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

To find the characteristic equation of A

The characteristic equation of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

Where, S_1= sum of main diagonal

=6+3+3=12

S_2 = sum of minor of main diagonal elements

$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$

= (9-1)+(18-4)+(18-4) = 8+14+14 = 36

S_3 = Det (A)=|A|

$= \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix} = 32$

Hence the characteristic equation is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

if λ=1; 1-12+36-32 ≠0

if λ=-1; -1-12-36-32≠0

if λ=2;8-42+72-32=0

By synthetic division

$$\begin{array}{r|rrrr} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \boxed{0} \end{array}$$

Therefore the λ=2 is a root

and other roots are given by $\lambda^2 - 10\lambda + 16 = 0$

$(\lambda-8)(\lambda-2) = 0$

$\lambda = 8, 2$

Therefore Eigenvalues are 8, 2, 2.

To find the Eigenvectors:

To get the Eigenvectors solve (A-λI) X=0

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \ldots\ldots \text{(A)}$$

Case (1): If λ = 8, then the equation (A) becomes

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(i.e) $-2x_1-2x_2+2x_3=0$ .... (1)

$-2x_1-5x_2-x_3=0$ ..... (2)

$2x_1-x_2-5x_3=0$ ..... (3)

Solving (1) and (2) by rule of cross multiplication, we get

	x_1	x_2	x_3
-2	2	-2	-2
-5	-1	-2	-5

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$

$$\Rightarrow \frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Hence the corresponding Eigenvector is $X_1=\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Case (2): If $\lambda = 2$ then the equation (A) becomes

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(i.e) $4x_1-2x_2+2x_3=0$ .... (4)

$-2x_1+x_2-x_3=0$ .... (5)

$2x_1-x_2+x_3=0$ .... (6)

Here (4), (5), (6) represents the same equation,

$$2x_1 - x_2 + x_3 = 0$$

If $x_1=0$ we get $-x_2+x_3=0$

$$-x_2 = -x_3$$

$$x_2 = x_3$$

$$\text{(i.e)} \ \frac{x_2}{1} = \frac{x_3}{1}$$

Hence the corresponding eigenvector is $X_2=\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Let $X_3=\begin{bmatrix} l \\ m \\ n \end{bmatrix}$ as x_3 is orthogonal to x_1 and x_2 since the given matrix is symmetric

$$[2 \ -1 \ 1]\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \ \text{ or } 2l-m+n=0 \qquad \ldots. \qquad (7)$$

$$[0 \ 1 \ 1]\begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0 \ \text{ or } 0l+m+n=0 \qquad \ldots. \qquad (8)$$

Solving (7) and (8) by rule of cross multiplication, we get

	l	m	n	
-1	1	2	-1	
1	1	0	1	

$$\frac{l}{-1-1} = \frac{m}{0-2} = \frac{n}{2-0}$$

$$\Rightarrow \frac{l}{-2} = \frac{m}{-2} = \frac{n}{2}$$

Hence the corresponding Eigenvector is $X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Result: The Eigenvalues are 8, 2, 2 and the Eigenvectors are $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

References:

Eigenvalues and Eigenvectors

Video Link: https://youtu.be/T8dPpuc8YN8

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Thank you